Beyond the Gaussian II : A Mathematical Experiment

Kazuyuki FUJII * and Hiroshi OIKE †

*Department of Mathematical Sciences

Yokohama City University

Yokohama, 236–0027

Japan

†Takado 85–5, Yamagata, 990–2464

Japan

Abstract

This is a sequel to the paper [K. Fujii : SIGMA **7** (2011), 022, 12 pages]. In this paper we treat a non-Gaussian integral based on a quartic polynomial and make a mathematical experiment by use of MATHEMATICA whether the integral is written in terms of its discriminant or not.

1 Introduction

The **Gaussian** is an abbreviation of all subjects related to the Gauss function $e^{-(px^2+qx+r)}$ like the Gaussian beam, Gaussian process, Gaussian noise, etc. It plays a fundamental role in Mathematics, Statistics, Physics and related disciplines.

*E-mail address : fujii@yokohama-cu.ac.jp

†E-mail address : oike@tea.ocn.ne.jp

It is generally conceived that any attempts to generalize the Gaussian results would meet formidable difficulties. Hoping to overcome this high wall of difficulties of going beyond the Gaussian in the near future, a first step was introduced in [1], [2].

This is simply one modest step to go beyond the Gaussian but it already reveals many obstacles related with the big challenge of going further beyond the Gaussian.

2 Cubic Case

In this section we treat the cubic case. In the paper [1][1] the following "formula" is reported

$$\int\int e^{-\left(ax^3+bx^2y+cxy^2+dy^3\right)}dxdy = \frac{1}{\sqrt[6]{-D}} \tag{1}$$

where D is the discriminant of the cubic equation

$$ax^3 + bx^2 + cx + d = 0, \tag{2}$$

and it is given by

$$D = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2. \tag{3}$$

The formula (1) is of course non-Gaussian. However, if we consider it in the framework of the real category then (1) is not correct because the left hand side diverges. In this paper we treat only the real category, and so a, b, c, d, x, y are real numbers.

Formally, by performing the change of variable $x = t\rho,\ y = \rho$ for (1) we have

$$\begin{aligned} \text{LHS of (1)} &= \int\int e^{-\rho^3\left(at^3+bt^2+ct+d\right)}|\rho|dtd\rho \\ &= \int\left\{\int e^{-\left(at^3+bt^2+ct+d\right)\rho^3}|\rho|d\rho\right\}dt \\ &= \int|\sigma|e^{-\sigma^3}d\sigma\int\frac{1}{|\sqrt[3]{(at^3+bt^2+ct+d)}|\sqrt[3]{(at^3+bt^2+ct+d)}}dt \end{aligned}$$

by the change of variable $\sigma = \sqrt[3]{at^3+bt^2+ct+d}\ \rho$.

The divergence comes from

$$\int|\sigma|e^{-\sigma^3}d\sigma,$$

[1] it is not easy for non–experts to understand this paper correctly

while the main part is

$$\int \frac{1}{|\sqrt[3]{(ax^3+bx^2+cx+d)}|\sqrt[3]{(ax^3+bx^2+cx+d)}}dx$$

under the change $t \to x$. As a kind of renormalization the integral may be defined like

$$\ddagger \int\int_{\mathbf{R}^2} e^{-(ax^3+bx^2y+cxy^2+dy^3)}dxdy \ddagger = \int_{\mathbf{R}} \frac{1}{|\sqrt[3]{(ax^3+bx^2+cx+d)}|\sqrt[3]{(ax^3+bx^2+cx+d)}}dx.$$

However, the right hand side lacks proper symmetry. If we set

$$F(a,b,c,d) = \int\int_{D_R} e^{-\left(ax^3+bx^2y+cxy^2+dy^3\right)}dxdy \tag{4}$$

where $D_R = [-R,R] \times [-R,R]$ $(R \gg 0)$, then it is easy to see

$$F(-a,-b,-c,-d) = F(a,b,c,d).$$

Namely, F is invariant under $\mathbf{Z}_2$–action. This symmetry is important and must be kept even in the renormalization process. The right hand side in the "definition" above is clearly not invariant. Therefore, by modifying it slightly we reach the renormalized integral

Definition

$$\ddagger \int\int_{\mathbf{R}^2} e^{-(ax^3+bx^2y+cxy^2+dy^3)}dxdy \ddagger = \int_{\mathbf{R}} \frac{1}{\sqrt[3]{(ax^3+bx^2+cx+d)^2}}dx. \tag{5}$$

The Gamma–function $\Gamma(p)$ is defined by

$$\Gamma(p) = \int_0^\infty e^{-x}x^{p-1}dx \quad (p>0) \tag{6}$$

and the Beta–function $B(p,q)$ is

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx \quad (p,\ q>0). \tag{7}$$

Then the result in [2] is

Formula

(I) For $D < 0$

$$\int_{\mathbf{R}} \frac{1}{\sqrt[3]{(ax^3+bx^2+cx+d)^2}} dx = \frac{C_-}{\sqrt[6]{-D}} \tag{8}$$

where

$$C_- = \sqrt[3]{2} B(\frac{1}{2}, \frac{1}{6}).$$

(II) For $D > 0$

$$\int_{\mathbf{R}} \frac{1}{\sqrt[3]{(ax^3+bx^2+cx+d)^2}} dx = \frac{C_+}{\sqrt[6]{D}} \tag{9}$$

where

$$C_+ = 3B(\frac{1}{3}, \frac{1}{3}).$$

(III) C_- and C_+ are related by $C_+ = \sqrt{3} C_-$ through the identity

$$\sqrt{3} B(\frac{1}{3}, \frac{1}{3}) = \sqrt[3]{2} B(\frac{1}{2}, \frac{1}{6}). \tag{10}$$

See [2] in detail. Our result shows that the integral depends on the sign of D. This formula has been conjectured by Morozov and Shakirov [1] in a different context.

A comment is in order. If we treat the Gaussian case (: $e^{-(ax^2+bxy+cy^2)}$) then the integral is reduced to

$$\int_{\mathbf{R}} \frac{1}{ax^2+bx+c} dx = \frac{2\pi}{\sqrt{-D}} = \frac{2B(\frac{1}{2}, \frac{1}{2})}{\sqrt{-D}} \tag{11}$$

if $a > 0$ and $D = b^2 - 4ac < 0$. Because

$$\pi = \frac{\sqrt{\pi}\sqrt{\pi}}{1} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = B(\frac{1}{2}, \frac{1}{2}).$$

Let us check whether the renormalized integral (5) is reasonable or not by making use of the results.

For the integral (4) it is easy to see

$$\begin{aligned}
\left(\frac{\partial}{\partial a}\frac{\partial}{\partial d} - \frac{\partial}{\partial b}\frac{\partial}{\partial c}\right) F(a,b,c,d) &= \int\int_{D_R} (x^3 \cdot y^3 - x^2 y \cdot xy^2) e^{-\left(ax^3+bx^2y+cxy^2+dy^3\right)} dxdy = 0, \\
\left(\frac{\partial}{\partial b}\frac{\partial}{\partial b} - \frac{\partial}{\partial a}\frac{\partial}{\partial c}\right) F(a,b,c,d) &= \int\int_{D_R} (x^2 y \cdot x^2 y - x^3 \cdot xy^2) e^{-\left(ax^3+bx^2y+cxy^2+dy^3\right)} dxdy = 0, \\
\left(\frac{\partial}{\partial c}\frac{\partial}{\partial c} - \frac{\partial}{\partial b}\frac{\partial}{\partial d}\right) F(a,b,c,d) &= \int\int_{D_R} (xy^2 \cdot xy^2 - x^2 y \cdot y^3) e^{-\left(ax^3+bx^2y+cxy^2+dy^3\right)} dxdy = 0.
\end{aligned} \tag{12}$$

On the other hand, if we set

$$\mathcal{F}(a,b,c,d)=\int_{\mathbf{R}}\frac{1}{\sqrt[3]{(ax^3+bx^2+cx+d)^2}}dx=\frac{C_{\pm}}{\sqrt[6]{\pm D}},$$

then we can also verify the same relations

$$\begin{aligned}
\left(\frac{\partial}{\partial a}\frac{\partial}{\partial d}-\frac{\partial}{\partial b}\frac{\partial}{\partial c}\right)\mathcal{F}(a,b,c,d)=0,\\
\left(\frac{\partial}{\partial b}\frac{\partial}{\partial b}-\frac{\partial}{\partial a}\frac{\partial}{\partial c}\right)\mathcal{F}(a,b,c,d)=0,\\
\left(\frac{\partial}{\partial c}\frac{\partial}{\partial c}-\frac{\partial}{\partial b}\frac{\partial}{\partial d}\right)\mathcal{F}(a,b,c,d)=0.
\end{aligned} \tag{13}$$

by use of MATHEMATICA.

Therefore we can say that the definition (5) is not so bad (maybe, good).

3 Quartic Case : Mathematical Experiment

In this section we treat the quartic case. The discriminant of the quartic equation

$$ax^4+bx^3+cx^2+dx+e=0 \tag{14}$$

is given by

$$\begin{aligned}
D &= 256a^3e^3-4b^3d^3-27a^2d^4-27b^4e^2-128a^2c^2e^2+b^2c^2d^2+16ac^4e\\
&\quad -4ac^3d^2-4b^2c^3e+144a^2cd^2e-6ab^2d^2e+144ab^2ce^2-192a^2bde^2\\
&\quad +18abcd^3+18b^3cde-80abc^2de.
\end{aligned} \tag{15}$$

See for example [2].

Here we consider a non–Gaussian integral

$$\mathcal{F}\equiv\mathcal{F}(a,b,c,d,e)=\int\int e^{-\left(ax^4+bx^3y+cx^2y^2+dxy^3+ey^4\right)}dxdy \tag{16}$$

and study whether this integral can be written in terms of its discriminant (15) or not like the formula in the preceding section. The conclusion is **negative**, while we have very interesting mathematical "phenomena" stated in the following.

By use of the same change of variable $x = t\rho,\ y = \rho$ in the cubic case we have

$$\mathcal{F} = \int\int e^{-\rho^4\left(at^4+bt^3+ct^2+dt+e\right)}|\rho|dtd\rho = \int\left\{\int e^{-\left(at^4+bt^3+ct^2+dt+e\right)\rho^4}|\rho|d\rho\right\}dt$$

and the change of variable $\sigma = \sqrt[4]{at^4+bt^3+ct^2+dt+e}\,\rho$ $(at^4+bt^3+ct^2+dt+e>0)$ gives

$$\begin{aligned}\mathcal{F} &= \int|\sigma|e^{-\sigma^4}d\sigma\int\frac{1}{\sqrt[4]{(at^4+bt^3+ct^2+dt+e)^2}}dt \\ &= \frac{\sqrt{\pi}}{2}\int_{\mathbf{R}}\frac{1}{\sqrt[4]{(ax^4+bx^3+cx^2+dx+e)^2}}dx \qquad (17)\end{aligned}$$

because

$$\int_{-\infty}^{\infty}|\sigma|e^{-\sigma^4}d\sigma = 2\int_0^{\infty}\sigma e^{-\sigma^4}d\sigma = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Problem I How can we calculate (17) ?

There is no method to calculate at the present time, so we make a mathematical experiment by use of MATHEMATICA. From the lesson of quadratic and cubic cases it may conjecture

$$\mathcal{F} = \frac{C}{\sqrt[12]{-D}} \quad (D<0,\ a>0) \qquad (18)$$

where C is some constant and $12 = 4\times 3$.

For the integral of exponent (16) we have a system of differential equations

$$\begin{aligned}\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c}-\frac{\partial^2}{\partial b^2}\right)\mathcal{F} &= 0, \\ \left(\frac{\partial}{\partial a}\frac{\partial}{\partial d}-\frac{\partial}{\partial b}\frac{\partial}{\partial c}\right)\mathcal{F} &= 0, \\ \left(\frac{\partial}{\partial a}\frac{\partial}{\partial e}-\frac{\partial^2}{\partial c^2}\right)\mathcal{F} &= 0, \\ \left(\frac{\partial}{\partial b}\frac{\partial}{\partial d}-\frac{\partial^2}{\partial c^2}\right)\mathcal{F} &= 0, \qquad (19) \\ \left(\frac{\partial}{\partial b}\frac{\partial}{\partial e}-\frac{\partial}{\partial c}\frac{\partial}{\partial d}\right)\mathcal{F} &= 0, \\ \left(\frac{\partial}{\partial c}\frac{\partial}{\partial e}-\frac{\partial^2}{\partial d^2}\right)\mathcal{F} &= 0.\end{aligned}$$

The proof is straightforward. For example,

$$\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c}-\frac{\partial^2}{\partial b^2}\right)\mathcal{F} = \int\int\left\{x^4\cdot x^2y^2-(x^3y)^2\right\}e^{-\left(ax^4+bx^3y+cx^2y^2+dxy^3+ey^4\right)}dxdy = 0.$$

On the other hand, if we set

$$\widetilde{\mathcal{F}} = \frac{C}{\sqrt[12]{-D}} \tag{20}$$

then we have

$$\begin{aligned}
\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c} - \frac{\partial^2}{\partial b^2}\right)\widetilde{\mathcal{F}} &= -C\frac{(c^2-3bd+12ae)(3d^2-8ce)}{36(-D)^{13/12}}, \\
\left(\frac{\partial}{\partial a}\frac{\partial}{\partial d} - \frac{\partial}{\partial b}\frac{\partial}{\partial c}\right)\widetilde{\mathcal{F}} &= C\frac{(c^2-3bd+12ae)(cd-6be)}{18(-D)^{13/12}}, \\
\left(\frac{\partial}{\partial a}\frac{\partial}{\partial e} - \frac{\partial^2}{\partial c^2}\right)\widetilde{\mathcal{F}} &= -C\frac{(c^2-3bd+12ae)(c^2-2bd-4ae)}{9(-D)^{13/12}}, \\
\left(\frac{\partial}{\partial b}\frac{\partial}{\partial d} - \frac{\partial^2}{\partial c^2}\right)\widetilde{\mathcal{F}} &= C\frac{(c^2-3bd+12ae)(16ae-bd)}{36(-D)^{13/12}}, \\
\left(\frac{\partial}{\partial b}\frac{\partial}{\partial e} - \frac{\partial}{\partial c}\frac{\partial}{\partial d}\right)\widetilde{\mathcal{F}} &= -C\frac{(c^2-3bd+12ae)(6ad-bc)}{18(-D)^{13/12}}, \\
\left(\frac{\partial}{\partial c}\frac{\partial}{\partial e} - \frac{\partial^2}{\partial d^2}\right)\widetilde{\mathcal{F}} &= -C\frac{(c^2-3bd+12ae)(3b^2-8ac)}{36(-D)^{13/12}}
\end{aligned} \tag{21}$$

by use of MATHEMATICA (verification by hand is very tough).

As a result $\mathcal{F} \neq \widetilde{\mathcal{F}}$. However, from (21) we obtain an interesting quantity

$$E \equiv c^2 - 3bd + 12ae \iff ax^4 + bx^3 + cx^2 + dx + e. \tag{22}$$

It is not clear at the present time what E is, so we present

Problem II Make the property of E clear.

We believe that E will play an important role in the calculation.

In last, we note some interesting fact. If we set

$$\widehat{\mathcal{F}} = \frac{C}{\sqrt[12]{D}} \quad (D > 0)$$

then the same relations (21) hold apart from the sign

$$
\begin{aligned}
\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c}-\frac{\partial^2}{\partial b^2}\right)\widehat{\mathcal{F}} &= C\frac{(c^2-3bd+12ae)(3d^2-8ce)}{36D^{13/12}},\\
\left(\frac{\partial}{\partial a}\frac{\partial}{\partial d}-\frac{\partial}{\partial b}\frac{\partial}{\partial c}\right)\widehat{\mathcal{F}} &= -C\frac{(c^2-3bd+12ae)(cd-6be)}{18D^{13/12}},\\
\left(\frac{\partial}{\partial a}\frac{\partial}{\partial e}-\frac{\partial^2}{\partial c^2}\right)\widehat{\mathcal{F}} &= C\frac{(c^2-3bd+12ae)(c^2-2bd-4ae)}{9D^{13/12}},\\
\left(\frac{\partial}{\partial b}\frac{\partial}{\partial d}-\frac{\partial^2}{\partial c^2}\right)\widehat{\mathcal{F}} &= -C\frac{(c^2-3bd+12ae)(16ae-bd)}{36D^{13/12}},\\
\left(\frac{\partial}{\partial b}\frac{\partial}{\partial e}-\frac{\partial}{\partial c}\frac{\partial}{\partial d}\right)\widehat{\mathcal{F}} &= C\frac{(c^2-3bd+12ae)(6ad-bc)}{18D^{13/12}},\\
\left(\frac{\partial}{\partial c}\frac{\partial}{\partial e}-\frac{\partial^2}{\partial d^2}\right)\widehat{\mathcal{F}} &= C\frac{(c^2-3bd+12ae)(3b^2-8ac)}{36D^{13/12}}.
\end{aligned}
$$

These "phenomena" are interesting enough and worth studying in detail.

4 Concluding Remarks

In this note we treated a non-Gaussian integral based on a quartic polynomial and made a mathematical experiment by use of MATHEMATICA. Though our work is far from obtaining an explicit value of the integral, we found an interesting quantity E. In order to make it clear hard work will be needed.

See [3], [4] as recent results on this topic and [5] as a general introduction to non–linear algebras. We expect young mathematicians or mathematical physicists to take part in this fascinating topic.

Acknowledgment.

The author wishes to thank Ryu Sasaki for helpful comments and suggestions.

References

[1] A. Morozov and Sh. Shakirov : Introduction to Integral Discriminants, JHEP 0912:002, 2009, arXiv:0903.2595 [math-ph].

[2] K. Fujii : Beyond the Gaussian : SIGMA **7** (2011), 022, 12 pages, arXiv:0905.1363.

[3] A. Morozov and Sh. Shakirov : New and Old Results in Resultant Theory, arXiv:0911.5278 [math-ph].

[4] A. Stoyanovsky : On integral of exponent of a homogeneous polynomial, arXiv:1103.0514 [math.AG].

[5] V. Dolotin and A. Morozov : Introduction to Non–Linear Algebra, 2007, World Scientific, hep-th/0609022.